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MODEL OF SELF-ORGANIZATION OF AN ENSEMBLE OF CRACKS RADIATING SOUND

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A model of self-organization of cracks arising in a rock specimen (granite) compressed by a press is proposed. The model is based on the assumption of acoustic wave interaction between the cracks. To construct the model of self-organization of cracks, solutions of the Fokker–Planck equation are used. The experimentally observed spontaneous increase in the activity of acoustic emission, spatial and temporal clusterization, and formation of a fractal structure in rock specimens under constant and slowly varying loads are explained.

Self-organized systems have recently gained much attention in physics of nonlinear processes and phenomena [1–3] and in geophysics, particularly, in studying the earthquake physics [4–6]. Most of the studies are based on the Barridge–Knopoff seismicity model and the Gutenberg–Richter law.

Despite long-standing efforts of researchers, the nature of an earthquake is still not understood. One line of investigations performed in this field is the laboratory simulation of the processes that occur in rock specimens compressed by means of a press. Interesting results were obtained which, to the author's knowledge, have not yet been explained. It was found that, after a specimen is subjected to constant pressure within approximately 50 h, the intensity of acoustic emission (AE) increases substantially, and the velocity of sound and amplitude of sounding signals change. Upon reaching a maximum, AE decreases almost to the initial level (prior to its increase). At the same time, the velocity and amplitude of the sounding signals do not revert to the initial level [7]. Similar results were obtained in [8] where nonstationary acoustic emission was studied. In contrast to [7], the velocity and amplitude of the sounding signal were not measured in [8]. Zhurkov et al. [8] found that the rate of crack formation increased spontaneously by a factor of 10 to 15 compared to the background level and then it decreased abruptly. The authors of [8] did not explain this phenomenon and considered that it was unlikely that the activity of crack formation increased abruptly in bulk of the loaded specimen. Apparently, this is possible only in a certain region of the specimen that is special for some reasons. A more detailed analysis of the kinetics of this local failure is required to explain why foreshocks, aftershocks, and possibly, some earthquake precursors occur.

Lei et al. [9] studied the distribution of AE hypocenters in a granite specimen under increasing three-dimensional compression. Three stages of deformation were observed. At the first stage, the location of AE hypocenters is random; at the second stage, they unite into groups (clusters); at the third stage which is prior to failure, the AE hypocenters form an ensemble (nucleation). In granite specimens (of the INADA type), the AE hypocenters unite into spatial–temporal fractals, and as the compression of the specimen is increased, which leads to failure, the dimension of the fractals d decreases from 2.8 at the second stage to 2.0 at the third stage [9]. (In a granite specimen of the OSHIMA type, self-organization of AE and formation of fractals do not occur.)

To understand the phenomenon of self-organization of acoustically active cracks, it is necessary to answer the following questions: Why does AE increase spontaneously and why and how are the radiating cracks clustered into fractals?

The essence of the model of self-organization of cracks is to find a possible mechanism of coherent interaction between the cracks by means of exchanging the acoustic waves emitted by the cracks. It is known that, when a

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crack is opening, it emits an AE pulse, which carries a certain fraction of energy; diffracting on a microcrack, the pulse transfers the energy, thus making the microcrack open. The crack grows and emits an acoustic pulse, which interacts with next crack, etc. In this case, cracks occur in an avalanche-like fashion, and their acoustic pulses are summed coherently, which amplifies the sound wave [7] and leads to occurrence of fractal structures similar to those described in [3].

The intensity I_λ of the sound wave emitted by a crack has the form

$$I_\lambda = \frac{1}{\tau_\lambda} \int \sigma_\lambda v_\lambda \, dt,$$

where σ_λ is the stress in the wave, v_λ is the particle-oscillation velocity in the medium, and τ_λ is the duration of the acoustic pulse. The quantity I_λ can also be written in the form adopted in laser physics:

$$I(x,t) = \frac{1}{\tau_\lambda}\Big(i \sum 2\pi h\omega_\lambda\Big) \exp{(ik_\lambda x)} b_\lambda.$$

Here $h\omega_\lambda$ is the energy of a sound "quantum," λ is the subscript denoting the mode of acoustic oscillations, k_λ is the wavenumber, b_λ is the complex amplitude (dimensionless quantity), and x is the coordinate.

Generally, self-organized structures are described by the Fokker–Planck or Ginzburg–Landau differential equations. We show that the Fokker–Planck equation can be used in the model proposed. We assume that, as in laser physics, one can introduce a two-level system in the acoustic model. Microcracks refer to the higher energy level and open cracks to the lower energy level. Upon opening of a crack, dipole radiation of an energy "quantum" occurs, and the system passes to the lower level. We introduce the concepts of the population inversion ε_μ and dipole moment of the crack α_μ. Here we use the dimensionless parameter $\alpha_\mu \sim l(t)/l$ [$l(t)$ is the crack growing up to the length l]. The physical meaning of the dipole moment of a crack is that the crack is considered as a radiator of the Hertz dipole type whose dipole moment is equal to the product of the current element and the length element. The inversion is given by $\varepsilon_\mu = N_0 - N$, where N_0 is the number of microcracks and N is the number of open cracks.

According to the Griffith criterion, the rate of crack opening is

$$u = \frac{dl}{dt} = B\Big(\frac{E}{\rho}\Big)^{1/2}\Big(1 - \frac{\Delta s}{\Delta w}\Big)^{1/2},$$

where B is a constant, E is Young's modulus, ρ is the density of the medium, Δs is the increase in the surface energy of the medium due to the crack growth, and Δw is the energy spent on increasing the crack length.

The crack opening is accompanied by radiation of an acoustic wave characterized by the stress σ_λ:

$$\sigma_\lambda = i\rho\omega_\lambda 2\pi l_\lambda^3 u_\lambda [(ik_\lambda x - 1)/(4\pi x^2)] \exp{(ik_\lambda x)} \cos\theta_\lambda.$$

Here $\omega_\lambda \sim 1/\tau_\lambda$ is the "frequency" of the radiated wave, $\tau_\lambda = l_\lambda/u_\lambda$ is the duration of the crack opening, x is the distance from the radiator (crack) to the observation point, and θ_λ is the angle of wave propagation relative to the crack-opening direction. The mode amplitude is $b_\lambda = \sigma_\lambda(t)/\sigma_\lambda$.

As the wave propagates in a medium with microcracks, it diffracts on one of the microcracks and transfers a part of the specific power $d\sigma/dt$ to it, thus making a contribution to its opening and subsequent growth [10]:

$$\frac{d\sigma}{dt} = \frac{2\mu_0 u}{(2\pi x)^{1/2}}\Big(\frac{\omega}{v_s}\Big)^{1/2} \sin\frac{\theta_\lambda}{2} \exp\Big[-i\Big(\omega t + \frac{\pi}{4}\Big)\Big].$$

Here μ_0 is the shear modulus and $v_s = (\mu_0/\rho)^{1/2}$.

An increase in the crack size is expressed in terms of the change in its dipole moment α_μ [1]:

$$\frac{\partial \alpha_\mu}{\partial t} = (-i\omega - \zeta)\alpha_\mu + \sum g_{\mu\lambda} b_\lambda \varepsilon_\mu + \Gamma_\mu(t).$$

Here ω is the frequency, ζ is the width of the emission line (this quantity has a similar meaning in optics), in this case, these are close quantities, and $\Gamma_\mu(t)$ are the fluctuating forces. If $\varepsilon_\mu = N_0 - N > 0$, the sound wave is amplified; if $\varepsilon_\mu < 0$, the wave is absorbed.

In a stressed medium in a stationary state, the principle of detailed equilibrium (principle of reversibility at the microlevel)

$$N_0\omega_0 = N\omega, \qquad g_0\omega_0 = g\omega$$

holds, where $\omega_0 \sim 1/\tau_0$ [τ_0 is the time it takes to "heal" the crack (usually, $\tau_0 \gg \tau$)], and g and g_0 are constants of interaction between the wave and the crack (these quantities have a similar meaning in optics). It follows that $\omega_0 \ll \omega$ and $N_0 \gg N$. Thus, the system of cracks is characterized by an inversion ε_μ whose derivative has the form

$$\frac{\partial \varepsilon_\mu}{\partial t} = \varepsilon_\mu \gamma + i \sum g_{\mu\lambda}\alpha_\mu b_\lambda + \Gamma_\mu(t),$$

where γ is a quantity reciprocal to the time of inversion relaxation to equilibrium.

In a stressed medium under a load close to the failure load, the Coulomb–Mohr failure criterion is valid. According to this criterion, the maximum shear stresses in a specimen act in the planes inclined at an angle β with respect to the axis of loading, which is close to 45°. Let $\beta = 45° - \varphi/2$ ($\tan\varphi = \nu$, where ν is the coefficient of internal friction). The magnitude of the angle β is independent of the strength of the material.

The essence of the adiabatic approach [1] is that the relaxation time of a crack τ_μ is much smaller than the life time of a stable mode of the system T: $\tau_\mu \ll T$ ($\tau_\mu = l_\mu/u$ and $T = L/v_p$, where L is the dimension of the system). Owing to the presence of feedback in the system, mainly those cracks open and grow which are additionally loaded by the stress $\Delta\sigma$ as a result of diffraction of vibrations of stable modes on these cracks. This, in turn, leads to an increase in the density of the sound flux in a given direction.

The "quantum" of the sound wave radiated by one crack can be considered as a small parameter: $e = h\omega = (\sigma^2/E)l^3$ (σ is the stress in the medium). The acoustic equivalent of the Planck constant has the form $h_a = (\sigma^2/(uE))l^4$. We assume that microcracks of size $l_0 \approx 1$ μm refer to the upper energy level and open cracks of size $l \approx 100$ μm refer to the lower energy level. We give the upper limit of the size, since the cracks of larger size tend to grow further and cannot be "healed" after opening. The energy of a microcrack $e_0 = (\sigma^2/E)l_0^3$ is much smaller than that of an open crack; therefore, it may be ignored in estimating the magnitude of a sound "quantum" formed upon opening of the crack. If $\sigma = 3 \cdot 10^7$ N/m^2 and $E = 10^{10}$ N/m^2, then $e = 10^{-7}$ J (for a 100-μm crack). We assume that the rate of crack opening is $u \approx 10^5$ cm/sec, $\tau = 10^{-7}$ sec, and $\omega = 10^7$ sec^{-1}, then $h_a = 10^{-14}$ J $\cdot$ sec. (This value is about 20 orders of magnitude greater than the quantum Planck constant $h = 6.6 \cdot 10^{-34}$ J $\cdot$ sec.)

Crack nucleation is a random process whose probability does not depend on the prehistory of the system. It is generally agreed that these processes are the Markov (Poisson) processes. The probability that the system at the moment $t + \Delta t$ is in a state with the parameter lying on the interval $(q, q + dq)$ can be determined by the Smoluchowski integral equation

$$f(q, t + \Delta t) = \int f(q_0, t) g(q_0, q - q_0, \Delta t)\, dq,$$

where $g(q_0, q - q_0, \Delta t)$ is the probability that the system passes from the point q_0 to the point q in the time Δt. After standard transformations of this equation, we obtain the one-dimensional Fokker–Planck equation

$$\frac{\partial f(q,t)}{\partial t} = -\frac{\partial j}{\partial q} \qquad \Big(j = \frac{d(\gamma q f)}{dq} + \frac{1}{2} Q \frac{d^2 f}{dq^2}\Big).$$

Here $\eta q = K$ is the drift coefficient, Q is the diffusion coefficient, and η is the attenuation rate of the wave packet in the system. It is known that the solution of this equation corresponds to the presence of a self-organization mechanism in the system, which consists in the interaction of two transfer phenomena: drift and diffusion (percolation). In the model proposed, both processes occur.

The Fokker–Planck equation yields stationary solutions with the argument independent of time or time-dependent solutions that do not depend on the coordinate. We consider some known solutions of the equation, which was first proposed by Fokker and Planck in 1914 to describe the regularities of the distribution of the mean energy of an electric dipole rotating in a radiation field. It is noteworthy that this equation was originally supposed to describe physics of the interaction of a particle with radiation (field). It was found later that it can be used to explain many self-organization phenomena in the fields of physics, chemistry, biology, and sociology [1].

In the one-dimensional case, the stationary solution of the Fokker–Planck equation has the form [1]

$$f(q) = P \exp(-2V(q)/Q),$$

where $V(q) = -\int K(q)\,dq$ is the potential and P is the normalization factor.

Klimontovich [2] obtained another stationary solution of this equation:

$$f(q) = \exp[F_0 - (aq + (1/2)bq)/D].$$

Here F_0 is the free energy (analog of fluctuating forces), a is the feedback parameter ($a = 0$ corresponds to the beginning of generation), b is the nonlinearity parameter, and D is the Gaussian noise intensity.

For both solutions, the probability density functions have an exponential form, the exponent containing a "force" parameter, which characterizes the potential, energy, etc. From the physical viewpoint, the solution of the Fokker–Planck equation can be interpreted as a dependence of the probability of occurrence of a function with a certain potential on this potential. The higher this potential, the lower the probability of this solution.

Let us consider nonstationary solutions. The one-dimensional solution of the nonstationary (time-dependent) Fokker–Planck equation has the form

$$f(q,t) = (\pi a(t))^{-1/2} \exp[-(q - b/t)^2/a(t)],$$

where $a(t) = (Q/\alpha)(1 - \exp(-2\alpha t)) + a_0 \exp(-2\alpha t)$ and $b(t) = b_0 \exp(-\alpha t)$. As $a \to 0$ ($a_0 = 0$), the solution becomes the δ function. This solution implies that a nonstationary solution can occur in a dissipative self-organized system under certain conditions, for example (for an appropriate interpretation of the parameters that enter the Fokker–Planck equation), in the form of a solitary wave. As is shown in [1], the solution (in the form of a wave or a δ function) can be gradually extended and weaken in space and time or, vice versa, compressed and amplified. This solution explains spontaneous amplification of AE in a rock specimen under constant load.

The linearized Fokker–Planck equation yields

$$\frac{dq}{dt} = -\alpha q + \eta \Delta q + F.$$

Here α is an external parameter (its physical meaning is the current density).

In the one-dimensional case, the correlation function $\langle q(x',t')q(x,t)\rangle$ has the form $\langle q(x',t)q(x,t)\rangle = Q/(\alpha\eta)^{1/2} \exp(-(\alpha/\eta)^{1/2}|x' - x|)$ for $t' = t$. The coefficient of $|x' - x|$ in the exponent has a dimension reciprocal to the dimension of length. We denote the correlation length by $l_k = (\alpha/\eta)^{-1/2}$. Obviously, $l_k \to \infty$ as $\alpha \to 0$ and, on the contrary, the correlation length decreases as the current density increases.

The parameter $d = (\alpha/\gamma)^{1/2} l_k$ is the dimension of the fractal (cluster). In [9], d varies from 2.8 to 2.2. According to estimation of the experimental data of [9], $l_k \approx 1$ cm. The pulse-flux density of acoustic emission is $\alpha \approx 10$–100 cm$^{-2} \cdot$ sec^{-1}; hence, we obtain $\eta \approx 1$–10 sec^{-1}. The existing experimental data do not allow one to estimate the parameter η; therefore, it is difficult to say how the above estimate is close to the actual value. (We recall that the parameter η characterizes the attenuation of an acoustic wave.)

The above-considered solution of the Fokker–Planck equation suggests the possibility of formation of a fractal structure of radiating cracks. We note that Lei et al. [9] observed the phenomenon of spatial clusterization of cracks, which "shrinked" from the bulk of the specimen to a certain plane inclined at an angle of approximately 45° with respect to the loading direction.

Self-organization in dissipative (nonconservative) multiparameter structures is quite a common process characterized by an exponential dependence of probability of one event or another on its parameter. For such phenomena as earthquakes, solar flares, and cosmic rays, energy can play the role of this parameter. This parameter may be also the frequency of crack formation.

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